

A SHARP BOUND ON THE LEBESGUE CONSTANT FOR LEJA POINTS IN THE UNIT DISK

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ABSTRACT. We give a sharp bound for the Lebesgue constant associated to Leja sequences in the complex unit disk, confirming a conjecture made by Calvi and Phung [2].

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Let K be a compact set in the complex plane. Leja sequences are defined by arbitrarily fixing a first point $e_0 \in K$ then by recursively selecting e_k , $k = 1, 2, \dots$ such that

$$(1) \quad \prod_{j=0}^{k-1} |e_k - e_j| = \max_{z \in K} \prod_{j=0}^{k-1} |z - e_j|.$$

They were first studied by Erdei [6, page 78], then by Leja [7] who showed that the sequence $\left(\prod_{j=0}^{k-1} |e_k - e_j|\right)^{\frac{1}{k}}$ converges to the transfinite diameter of K .

Consider $\mathcal{C}(K)$ the space of continuous functions on K , endowed with the uniform norm. For any $f \in \mathcal{C}(K)$, the unique polynomial in the space Π_{k-1} of polynomials of degree at most $k-1$ which coincides with f on $E_k = (e_0, e_1, \dots, e_k)$ is the Lagrange interpolation polynomial defined by

$$L_{E_k}(f)(z) = \sum_{j=0}^{k-1} f(e_j) l_{j,E_k}(z)$$

where

$$l_{j,E_k}(z) = \prod_{i=0, i \neq j}^{k-1} \frac{z - e_i}{e_j - e_i}$$

are the fundamental Lagrange interpolation polynomials.

The norm of L_{E_k} as a continuous linear operator from $\mathcal{C}(K)$ into Π_{k-1} is the so-called Lebesgue constant

$$(2) \quad \Lambda_{E_k} := \sup_{\|f\| \leq 1} \|L_{E_k}(f)\| = \sup_{z \in K} |\lambda_{E_k}(z)|,$$

where

$$\lambda_{E_k}(z) := \sum_{j=0}^{k-1} |l_{j,E_k}(z)|.$$

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The Lebesgue constant plays a crucial role in polynomial interpolation. The inequality

$$\|L_{E_k}(f) - f\|_K \leq (\Lambda_{E_k} + 1) \inf_{p \in \Pi_{k-1}} \|f - p\|,$$

shows that it measures how close the interpolant is to the best polynomial approximant of a function. It also measures the stability of the Lagrange interpolation. We refer the reader to [8] for more details and many interesting properties on the Lebesgue constant.

In this paper, we are interested in finding an optimal bound for the Lebesgue constant associated with Leja points in the case where K is the complex unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| \leq 1\}$.

We consider Leja sequences $(e_k)_{k \geq 0}$ initiated at $e_0 \in \partial\mathcal{U} = \{|z| = 1\}$. There is no loss of generality in assuming that $e_0 = 1$ since any Leja sequence is the product by e_0 of a Leja sequence initiated at 1. We speak of Leja sequences rather than of a Leja sequence because at each step there may be more than one point e_k satisfying (1). Note that by the maximum principle, $|e_k| = 1$, $k = 1, 2, \dots$.

Any finite sequence $E_k = (e_0, \dots, e_k)$ where e_0, \dots, e_k are defined by (1) is called a k -Leja section.

Leja sequences of the disk were explicitly described by Bialaz-Ciez and Calvi [1]. They showed that for any Leja sequence of the disk, initiated at $e_0 = 1$, the underlying set of the 2^n -th section consists of the 2^n -th roots of the unity. The 2^{n+1} -th section is

$$(E_{2n}, \rho F_{2n})$$

where ρ is any 2^n -th root of -1 and $F_{2n} = (f_0, \dots, f_{2n-1})$ is a 2^n -Leja section with $f_0 = 1$, with the notation

$$(E_k, F_j) := (e_0, \dots, e_{k-1}, f_0, \dots, f_{j-1}).$$

A natural Leja sequence is then inductively constructed as follows :

$$E_1 = (1), \quad (E_{2^n}, e^{\frac{i\pi}{2^n}} E_{2^n}).$$

It was shown in [1] that this particular sequence is defined by

$$(3) \quad e_k = \exp(i\pi \sum_{j=0}^s 2^{-p_j})$$

where $k \geq 1$ is expanded in the binary form

$$(4) \quad k = 2^{p_0} + \dots + 2^{p_s}, \quad 0 \leq p_0 < p_1 < \dots < p_s.$$

Let us now recall the previous results about the bounds of the Lebesgue constant for Leja points in the disk.

Calvi and Phung [2] showed, for any k -Leja section $E_k = (e_0, \dots, e_k)$, that

$$\Lambda_{E_k} = O(k \ln k).$$

They also showed that

$$\Lambda_{E_{2^n-1}} = 2^n - 1, \quad n \geq 1$$

so it cannot grow slower than k . They conjectured the following :

$$(5) \quad \Lambda_{E_k} \leq k$$

Chkifa [3] gave the sharper bound :

$$\Lambda_{E_k} \leq 2k.$$

In the same direction, Irigoyen [5] showed that there is a uniform bound for the fundamental Lagrange interpolation polynomials, namely :

$$\sup_{k>j\geq 0} \left(\sup_{z \in \mathcal{U}} |l_{j,E_k}(z)| \right) \leq \pi \exp(3\pi).$$

In a recent work Chkifa [4] introduced the so-called "quadratic" Lebesgue function

$$\lambda_{E_k,2}(z) = \left(\sum_{j=0}^{k-1} |l_{j,E_k}(z)|^2 \right)^{1/2}$$

and the "quadratic" Lebesgue constant

$$\Lambda_{E_k,2} := \max_{z \in \mathcal{U}} \lambda_{E_k,2}(z),$$

which turned out to be quite efficient tools. Their advantage is that they strongly exploit the binary structure of the Leja sequences. It was proved in [4] that for any k expanded in the form (4), the following inequalities hold :

$$(6) \quad \sqrt{2^{s+1} - 1} = \lambda_{E_k,2}(e_k) \leq \Lambda_{E_k,2} \leq \sqrt{3(2^{s+1} - 1)}$$

and by the Cauchy-Schwarz inequality,

$$\Lambda_{E_k} \leq \sqrt{3k(2^{s+1} - 1)},$$

In the present paper, we will follow the same approach as in [4]. Our estimate for the "quadratic" Lebesgue constant is given by next proposition.

Proposition 1. *For any $k \geq 1$ and any k -Leja section $E_k = (e_0, \dots, e_k)$ in the unit disk, we have :*

$$\Lambda_{E_k,2} \leq \sqrt{2^{-p_0} k},$$

where $k = 2^{p_0} + \dots + 2^{p_s}$ is expanded in the form (4).

Remark 1. *This a sharp bound : in the case where $k = 2^n - 1$, the proposition combined with the first inequality in (6) shows that we actually have*

$$(7) \quad \Lambda_{E_{2^n-1},2} = \sqrt{2^n - 1}$$

Now, by Proposition 1 and a straightforward application of the Cauchy-Schwarz inequality, we settle conjecture (5) :

Theorem 1. *For any $k \geq 1$ and any k -Leja section $E_k = (e_0, \dots, e_k)$ in the unit disk, the following inequality holds :*

$$\Lambda_{E_k} \leq 2^{-\frac{p_0}{2}} k,$$

where $k = 2^{p_0} + \dots + 2^{p_s}$ is expanded in the form (4).

2. PROOF OF PROPOSITION 1

It was proved in [3, Lemma 2.4] that for any $k \geq 0$ and any two k -Leja sections E_k and F_k of the unit disk, there exists $|\rho| = 1$ such that $F_k = \rho E_k$ (in the set sense). This implies that the Lebesgue constant of a k -Leja sequence of the disk only depends on k .

We may then assume, without loss of generality, that we are dealing with the particular Leja-sequence, that we will denote by $E = (e_k)_{k \geq 0}$, defined by (3).

We will use the simplified notations :

$$l_{j,k}(z) = \prod_{i=0, i \neq j}^{k-1} \frac{z - e_i}{e_j - e_i}, \quad 0 \leq j \leq k-1,$$

$$\lambda_{k,2}(z) = \left(\sum_{j=0}^{k-1} |l_{j,k}(z)|^2 \right)^2,$$

$$\Lambda_{k,2}(z) = \sup_{|z| \leq 1} \lambda_{k,2}(z) = \sup_{|z|=1} \lambda_{k,2}(z).$$

The last equality comes from the subharmonicity of $\lambda_{k,2}$ and the maximum principle.

Recall that, by definition of Leja sequences,

$$(8) \quad |l_{k-1,k}(z)| = \prod_{j=0}^{k-2} \frac{|z - e_j|}{|e_{k-1} - e_j|} \leq 1, \quad k \geq 2.$$

Exploiting the following relations satisfied by these Leja points :

$$(9) \quad e_{2j+1} = -e_{2j}, \quad e_{2j}^2 = e_{2j+1}^2 = e_j, \quad j \geq 0,$$

it was shown in [4] that :

$$\lambda_{2N,2}(z) = \lambda_{N,2}(z^2), \quad N \geq 0, \quad |z| \leq 1$$

and as consequences :

$$\Lambda_{2N,2} = \Lambda_{N,2}, \quad N \geq 1$$

$$\Lambda_{2^n,2} = \Lambda_{1,2} = 1, \quad n \geq 0.$$

To get an accurate estimate on the Lebesgue constant, we need a recursive formula for $\Lambda_{2N+1,2}$. We will use similar techniques as in [4] to show the following :

Lemma 1. *For all $N \geq 1$,*

$$\Lambda_{2N+1,2}^2 \leq \frac{1}{2} \Lambda_{N+1,2}^2 + 2\Lambda_{N,2}^2 + \frac{1}{2}.$$

Remark 2. *We may already deduce by induction from this lemma that, for all $n \geq 1$,*

$$\Lambda_{2^n-1,2}^2 \leq 2^n - 1.$$

Proof.

Let $|z| \leq 1$. Using the relations (9), we find

$$l_{2N,2N+1}(z) = \prod_{i=0}^{N-1} \frac{z^2 - e_i}{e_N - e_i} = l_{N,N+1}(z^2).$$

For $j = 0, 1, \dots, N-1$, we have

$$\begin{aligned} l_{2j,2N+1}(z) &= \frac{(z - e_{2N})(z - e_{2j+1})}{(e_{2j} - e_{2N})(e_{2j} - e_{2j+1})} \prod_{i=0, i \neq j}^{N-1} \frac{(z^2 - e_i^2)}{(e_j - e_i)} \\ &= \frac{(z - e_{2N})(z + e_{2j})(e_{2j} + e_{2N})}{2e_{2j}(e_j - e_N)} l_{j,N}(z^2). \end{aligned}$$

And similarly,

$$l_{2j+1,2N+1}(z) = \frac{(z - e_{2N})(z - e_{2j})(e_{2j} - e_{2N})}{2e_{2j}(e_j - e_N)} l_{j,N}(z^2).$$

Now, we use that :

$$|(z + e_{2j})(e_{2j} + e_{2N})|^2 + |(z - e_{2j})(e_{2j} - e_{2N})|^2 = 2|z + e_{2N}|^2 + 2|ze_{2N} + e_j|^2$$

and that

$$|((z - e_{2N})(ze_{2N} + e_j))|^2 \leq 2|z^2 - e_j|^2 + 2|e_j - e_N|^2.$$

We deduce the following :

$$\begin{aligned} |l_{2j,2N+1}(z)|^2 + |l_{2j+1,2N+1}(z)|^2 &\leq \frac{|z^2 - e_N|^2 + 2|z^2 - e_j|^2 + 2|e_j - e_N|^2}{2|e_j - e_N|^2} |l_{j,N}(z^2)|^2 \\ &= \frac{1}{2} |l_{j,N+1}(z^2)|^2 + |l_{N,N+1}(z^2)|^2 |l_{j,N}(e_N)|^2 + |l_{j,N}(z^2)|^2 \end{aligned}$$

We are now ready to estimate $\lambda_{2N+1,2}(z)$:

$$\lambda_{2N+1,2}^2(z) \leq \frac{1}{2} \lambda_{N+1,2}^2(z^2) + |l_{N,N+1}(z^2)|^2 \lambda_{N,2}^2(e_N) + \lambda_{N,2}^2(z^2) + \frac{1}{2} |l_{N,N+1}(z^2)|^2.$$

Applying inequality (8), the proof of the lemma is completed. □

Let us now define the sequence $(U_k)_{k \geq 1}$ by :

$$U_1 = 1, U_{2N} = U_N, U_{2N+1} = \frac{1}{2}U_{N+1} + 2U_N + \frac{1}{2}, \quad N \geq 1.$$

We easily verify by induction on $n \geq 0$ that for all $m \geq 0$:

$$(10) \quad U_{2^n m} = U_m;$$

$$(11) \quad U_{2^{n+1} m} = 2^{-n} U_{m+1} + 4(1 - 2^{-n}) U_m + 1 - 2^{-n}.$$

Thanks to Lemma 1, we have :

$$\Lambda_{k,2}^2 \leq U_k, \quad k \geq 1.$$

So the proof of Proposition 1 will be done if we prove that, for all $k = 2^{p_0} + 2^{p_1} + \dots + 2^{p_s}$ expanded as in (4),

$$U_k \leq 2^{-p_0} k,$$

or equivalently, that

$$(12) \quad \Delta_k := 2^{-p_0} k - U_k \geq 0.$$

Clearly, because of (10), we have

$$(13) \quad \Delta_{2^{p_0}+2^{p_1}+\dots+2^{p_s}} = \Delta_{1+2^{p_1-p_0}+\dots+2^{p_s-p_0}}.$$

When $s = 0$, we note that, for all $p_0 \geq 0$:

$$\Delta_{2^{p_0}} = \Delta_1 = 0.$$

Let us now deal with the case $s = 1$: Thanks to (11), for all $0 \leq p_0 < p_1$,

$$U_{1+2^{p_1-p_0}} = 2^{p_0-p_1} U_2 + 4(1 - 2^{p_0-p_1}) U_1 + 1 - 2^{p_0-p_1} = 4(1 - 2^{p_0-p_1}) + 1.$$

It follows that

$$(14) \quad \Delta_{2^{p_0}+2^{p_1}} = 2^{p_0-p_1} (2^{p_1-p_0} - 2)^2 \geq 0.$$

For the general case $s \geq 2$, (12) will be proved recursively on s in the following lemma :

Lemma 2. *For all $s \geq 1$ and for all $0 \leq p_0 < p_1 < \dots < p_s$:*

$$(15) \quad \begin{aligned} \Delta_{2^{p_0}+\dots+2^{p_s}} &= \sum_{j=1}^s 2^{j-p_j+p_0} (2^{p_j-p_{j-1}} - 2) [(3 \cdot 2^{j-1} - 1) \Delta_{2^{p_j}+\dots+2^{p_s}} \\ &\quad + (2^{j-2} (2^{p_j-p_{j-1}} - 2^2) + 1) (1 + 2^{p_{j+1}-p_j} + \dots + 2^{p_s-p_j})] \geq 0 \end{aligned}$$

Remark 3.

Δ_k measures how large $2^{-p_0} k$ is compared to $\Lambda_{k,2}^2$ in terms of the binary expansion of k .

Whenever $p_j - p_{j-1} = 1$, the factor $2^{p_j-p_{j-1}} - 2$ is a vanishing term and does not contribute to the sum. This lemma states in particular that :

$\Delta_k = 0$ if and only if $k = 2^{p_0} (2^n - 1)$ for some $p_0 \geq 0$ and $n \geq 1$.

For all the other numbers, the inequalities

$$\Lambda_k \leq \sqrt{k} \Lambda_{k,2} \leq \sqrt{k} \sqrt{2^{-p_0} k - \Delta_k}$$

are actually more accurate than the ones announced in Proposition 1 and Theorem 1.

Proof.

When $s = 1$, this corresponds to (14). Assume that $s \geq 2$ and that formula and inequality (15) are verified up to $s - 1$.

We apply (10) and (11) to get the next two formulas :

$$U_{2^{p_0}+\dots+2^{p_s}} = 2^{p_0-p_1} U_{2+2^{p_2-p_1}+\dots+2^{p_s-p_1}} + 4(1 - 2^{p_0-p_1}) U_{2^{p_1}+\dots+2^{p_s}} + 1 - 2^{p_0-p_1}.$$

$$U_{2^{p_1}+2^{p_2}+\dots+2^{p_s}} = \frac{1}{2} U_{2+2^{p_2-p_1}+\dots+2^{p_s-p_1}} + 2 U_{2^{p_2}+\dots+2^{p_s}} + \frac{1}{2}.$$

And we deduce that

$$U_{2^{p_0}+\dots+2^{p_s}} = 1 + 2^{p_0-p_1} \left[(2^{2+p_1-p_0} - 2)U_{2^{p_1}+\dots+2^{p_s}} - 4U_{2^{p_2}+\dots+2^{p_s}} - 2 \right].$$

In terms of the Δ'_k 's, the previous identity may be written as follows :

(16)

$$\begin{aligned} \Delta_{2^{p_0}+\dots+2^{p_s}} &= 2^{p_0-p_1} (2^{p_1-p_0} - 2) \left[4\Delta_{2^{p_1}+\dots+2^{p_s}} + (2^{p_1-p_0} - 2)(1 + 2^{p_2-p_1} + \dots + 2^{p_s-p_1}) \right] \\ &\quad + 2^{p_0-p_1} R_1(p_1, \dots, p_s), \end{aligned}$$

where

$$\begin{aligned} R_1(p_1, \dots, p_s) &:= -2(2^{p_2-p_1} - 2) \left[(1 + 2^{p_3-p_2} + \dots + 2^{p_s-p_2}) + 2^{1+p_1-p_2} \Delta_{2^{p_2}+\dots+2^{p_s}} \right] \\ &\quad + 6\Delta_{2^{p_1}+\dots+2^{p_s}} - 2^{3+p_1-p_2} \Delta_{2^{p_2}+\dots+2^{p_s}} \end{aligned}$$

After applying the recursive hypothesis (15) for ranks $s-1$ and $s-2$ on the second line, and setting apart the term corresponding to $j=1$ in $\Delta_{2^{p_1}+\dots+2^{p_s}}$, we obtain :

(17)

$$\begin{aligned} R_1(p_1, \dots, p_s) &:= 2^{2+p_1-p_2} (2^{p_2-p_1} - 2) \left[5\Delta_{2^{p_2}+\dots+2^{p_s}} + (2^{p_2-p_1} - 3)(1 + 2^{p_3-p_2} + \dots + 2^{p_s-p_2}) \right] \\ &\quad + R_2(p_2, \dots, p_s) \end{aligned}$$

where

$$\begin{aligned} R_2(p_2, \dots, p_s) &= 3 \sum_{j=2}^{s-1} 2^{1+j-p_{j+1}+p_1} (2^{p_{j+1}-p_j} - 2) \left[(3 \cdot 2^{j-1} - 1) \Delta_{2^{p_{j+1}}+\dots+2^{p_s}} \right. \\ &\quad \left. + (2^{j-2} (2^{p_{j+1}-p_j} - 2^2) + 1) (1 + 2^{p_{j+2}-p_{j+1}} + \dots + 2^{p_s-p_{j+1}}) \right] \\ &\quad - \sum_{j=1}^{s-2} 2^{3+j-p_{j+2}+p_1} (2^{p_{j+2}-p_{j+1}} - 2) \left[(3 \cdot 2^{j-1} - 1) \Delta_{2^{p_{j+2}}+\dots+2^{p_s}} \right. \\ &\quad \left. + (2^{j-2} (2^{p_{j+2}-p_{j+1}} - 2^2) + 1) (1 + 2^{p_{j+3}-p_{j+2}} + \dots + 2^{p_s-p_{j+2}}) \right]. \end{aligned}$$

Writing the last expression under one sum, we obtain :

$$\begin{aligned} (18) \quad R_2(p_2, \dots, p_s) &= \sum_{j=3}^s 2^{j-p_j+p_1} (2^{p_j-p_{j-1}} - 2) \left[(3 \cdot 2^{j-1} - 1) \Delta_{2^{p_j}+\dots+2^{p_s}} \right. \\ &\quad \left. + (2^{j-2} (2^{p_j-p_{j-1}} - 2^2) + 1) (1 + 2^{p_{j+1}-p_j} + \dots + 2^{p_s-p_j}) \right]. \end{aligned}$$

Finally, we put together equations (16), (17) and (18) and we have reached formula (15). \square

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